# Analytical Pricing of European Inflation Options

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Suppose that the underlying stochastic process $X_t$ satisfies the following stochastic differential equation:

$$dX_t = a(X_t)dt + b(X_t)dW_t, \tag{1}$$

where

$$a(X_t) = \alpha(G - X_t) \;\text{ and }\; b(X_t) = \sigma,$$

and $W_t$ is a Brownian motion. This is the Ornstein-Uhlenbeck process and is the process we use to model the evolution of inflation. (The Ornstein-Uhlenbeck process is described in [1].) To solve this equation, let $Y_t = X_t - G$. Then $dY_t = dX_t$ and

$$dY_t = -\alpha Y_t + \sigma dW_t.$$

Multiplying by $e^{\alpha t}$ leads to

$$e^{\alpha t} dY_t + \alpha e^{\alpha t} Y_t = \sigma e^{\alpha t} dW_t$$

and so

$$\int_0^t d(e^{\alpha s} Y_s) = \sigma \int_0^t e^{\alpha s}\, dW_s.$$

Therefore,

$$e^{\alpha t} Y_t = Y_0 + \sigma \int_0^t e^{\alpha s}\, dW_s,$$

namely

$$Y_t = e^{-\alpha t} Y_0 + \sigma \int_0^t e^{\alpha(s-t)}\, dW_s.$$

Therefore, the solution to equation (1) is

$$X_t = (X_0 - G)e^{-\alpha t} + G + \sigma \int_0^t e^{\alpha(s-t)}\, dW_s. \tag{2}$$

Now, the mean of $X_t$ is

$$\begin{aligned}
\mathbb{E}[X_t] &= \mathbb{E}\left[(X_0 - G)e^{-\alpha t} + G + \sigma \int_0^t e^{\alpha(s-t)}\, dW_s\right] \\
&= \mathbb{E}[(X_0 - G)e^{-\alpha t} + G] + \mathbb{E}\left[\sigma \int_0^t e^{\alpha(s-t)}\, dW_s\right] \\
&= (X_0 - G)e^{-\alpha t} + G + \sigma \mathbb{E}\left[\int_0^t e^{\alpha(s-t)}\, dW_s\right] \\
&= (X_0 - G)e^{-\alpha t} + G,
\end{aligned}$$

where we have used the martingale property of Itô integrals, namely, for an appropriate function $f$,

$$\mathbb{E}\left[\int_0^t f(W_s, s)\, dW_s\right] = 0.$$

Therefore, the variance of $X_t$ is

$$\begin{aligned}\mathrm{Var}(X_t) &:= \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] \\ &= \sigma^2 \mathbb{E}\left[\left(\int_0^t e^{\alpha(s-t)}\, dW_s\right)^2\right] \\ &= \sigma^2 \int_0^t e^{2\alpha(s-t)}\, ds \\ &= \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}),\end{aligned}$$

where we have used Itô isometry, namely, for an appropriate function $f$,

$$\mathbb{E}\left[\left(\int_0^t f(W_s, s)\, dW_s\right)^2\right] = \int_0^t \mathbb{E}[f(W_s, s)^2]\, ds.$$

Now, we know that

$$\sigma e^{-\alpha t} \int_0^t e^{\alpha s}\, ds = \frac{\sigma}{\alpha}(1 - e^{-\alpha t})$$

and that

$$W_T = \int_0^T dW_s.$$

Therefore, using equation (2), we can approximate the terminal value $X_T$, i.e., the value of the underlying at the option maturity date $T$, by

$$X_T = (X_0 - G)e^{-\alpha T} + G + \frac{\sigma(1 - e^{-\alpha T})}{\alpha} W_T.$$

The payoff of a call option with strike $K$ at maturity is given by

$$\pi_c(T, X_T) = \max\{X_T - K, 0\}.$$

Suppose $\mathbb{Q}$ is the risk-neutral measure under which the stochastic process $X_t$ is a $(\mathbb{Q}, \{\mathcal{F}_t\}_{t\geqslant 0})$-martingale, where $\{\mathcal{F}\}_{t\geqslant 0}$ is a filtration to which $X_t$ is adapted. Then, by the Fundamental Theorem of Asset Pricing [2], the price of the call option at time $t < T$ is given by

$$\mathbb{E}_{\mathbb{Q}}[\pi_c(t, X_T)|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}\left[e^{-r(T-t)} \max\left\{(X_0 - G)e^{-\alpha T} + G + \frac{\sigma(1 - e^{-\alpha T})}{\alpha} W_T - K, 0\right\}\middle| \mathcal{F}_t\right],$$

where $r$ is the risk-free rate of interest. Therefore, the value of the call option today, i.e., when $t = 0$, is

$$\mathbb{E}_{\mathbb{Q}}[\pi_c(0, X_T)] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[\max\left\{(X_0 - G)e^{-\alpha T} + G + \frac{\sigma(1 - e^{-\alpha T})}{\alpha} W_T - K, 0\right\}\right].$$

Since $W_T \sim N(0, \sqrt{T})$, we have $-W_T/\sqrt{T} \sim N(0, 1)$. To see this, note that the probability density function for the normal distribution with mean $\mu$ and variance $\sigma^2$ is

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}.$$

Substituting $\mu = 0$ and $\sigma = -\sqrt{T}/\sqrt{T}$ into this function gives $f(x) = (1/\sqrt{2\pi})e^{-x^2/2}$ and so $-W_T/\sqrt{T} \sim N(0,1)$. Writing $y = -W_T/\sqrt{T}$, we have $W_T = -y\sqrt{T}$ and clearly $y \sim N(0,1)$. Then

$$\mathbb{E}_{\mathbb{Q}}[\pi_c(0, X_T)] = \frac{e^{-rT}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \max\left\{ (X_0 - G)e^{-\alpha T} + G - K - \frac{\sigma y \sqrt{T}(1 - e^{-\alpha T})}{\alpha}, 0 \right\} e^{-y^2/2}\, dy.$$

To calculate this integral, we need to determine the range of $y$ for which the integrand is positive. We have

$$\begin{aligned} &\max\left\{ (X_0 - G)e^{-\alpha T} + G - K - \frac{\sigma y \sqrt{T}(1 - e^{-\alpha T})}{\alpha}, 0 \right\} \\ &= \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha} \max\left\{ \frac{\alpha[(X_0 - G)e^{-\alpha T} + G - K]}{\sigma\sqrt{T}(1 - e^{-\alpha T})} - y, 0 \right\}. \end{aligned}$$

Write

$$F(y) = \frac{\alpha\{(X_0 - G)e^{-\alpha T} + G - K\}}{\sigma\sqrt{T}(1 - e^{-\alpha T})} - y$$

and

$$f^* = \frac{\alpha\{(X_0 - G)e^{-\alpha T} + G - K\}}{\sigma\sqrt{T}(1 - e^{-\alpha T})}.$$

Then

$$F(y) > 0 \text{ iff } y < f^*$$

and so

$$\mathbb{E}_{\mathbb{Q}}[\pi_c(0, X_T)] = e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[ \left( (X_0 - G)e^{-\alpha T} + G - \frac{\sigma y \sqrt{T}(1 - e^{-\alpha T})}{\alpha} - K \right) \mathbf{1}_{y - f^* < 0} \right].$$

We, therefore, integrate over the range $(-\infty, f^*]$ to obtain

$$\begin{aligned} e^{rT}\mathbb{E}_{\mathbb{Q}}[\pi_c(0, X_T)] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{f^*} \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha} \left\{ \frac{\alpha[(X_0 - G)e^{-\alpha T} + G - K]}{\sigma\sqrt{T}(1 - e^{-\alpha T})} - y \right\} e^{-y^2/2}\, dy \\ &= \frac{[(X_0 - G)e^{-\alpha T} + G - K]}{\sqrt{2\pi}} \int_{-\infty}^{f^*} e^{-y^2/2}\, dy - \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha\sqrt{2\pi}} \int_{-\infty}^{f^*} y e^{-y^2/2}\, dy \\ &= \frac{[(X_0 - G)e^{-\alpha T} + G - K]}{\sqrt{2\pi}} \int_{-\infty}^{f^*} e^{-y^2/2}\, dy + \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha\sqrt{2\pi}} \int_{-\infty}^{f^*} \frac{d}{dy} e^{-y^2/2}\, dy \\ &= \frac{[(X_0 - G)e^{-\alpha T} + G - K]}{\sqrt{2\pi}} \int_{-\infty}^{f^*} e^{-y^2/2}\, dy + \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha} \frac{e^{-(f^*)^2/2}}{\sqrt{2\pi}} \\ &= [(X_0 - G)e^{-\alpha T} + G - K]N(f^*) + \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha} n(f^*), \end{aligned}$$

where

$$N(f^*) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{f^*} e^{-x^2/2}\, dx$$

and

$$n(f^*) := N'(f^*) = \frac{1}{\sqrt{2\pi}} e^{-(f^*)^2/2}.$$

Note that the equation

$$\int_{-\infty}^{f^*} \frac{d}{dy} e^{-y^2/2}\, dy = e^{-(f^*)^2/2}$$

is got by using

$$\lim_{y \to -\infty} e^{-y^2/2} = 0.$$

Finally, the value of the call option is then given by

$$\mathbb{E}_{\mathbb{Q}}[\pi_c(0, X_T)] = e^{-rT}[g\, N(f^*) + m\, n(f^*)],$$

where

$$g = (X_0 - G)e^{-\alpha T} + G - K \quad \text{and} \quad m = \frac{\sigma\sqrt{T}(1 - e^{-\alpha T})}{\alpha}.$$

A similar analysis can be undertaken to value a put option, where the payoff is given by

$$\pi_p(T, X_T) = \max\{K - X_T, 0\}.$$

Therefore, the value of a put option is given by

$$\mathbb{E}_{\mathbb{Q}}[\pi_p(0, X_T)] = e^{-rT}[\hat{g}\, N(\hat{f}^*) + m\, n(\hat{f}^*)],$$

where

$$\hat{f}^* = \frac{\alpha\{K - (X_0 - G)e^{-\alpha T} - G\}}{\sigma\sqrt{T}(1 - e^{-\alpha T})} \quad \text{and} \quad \hat{g} = K - (X_0 - G)e^{-\alpha T} - G.$$

The Greek sensitivities delta, vega, and gamma of the call option price can be computed as follows:

$$\begin{aligned}
\Delta &:= \frac{\partial}{\partial X_0}\pi_c(0, X_T) = e^{-\alpha T} N(f^*), \\
\Psi &:= \frac{\partial}{\partial \sigma}\pi_c(0, X_T) = \frac{\sqrt{T}(1 - e^{-\alpha T})}{\alpha} n(f^*), \\
\Gamma &:= \frac{\partial^2}{\partial X_0^2}\pi_c(0, X_T) \\
&= \frac{\partial}{\partial X_0} e^{-\alpha T} N(f^*) \\
&= e^{-\alpha T} \frac{\partial f^*}{\partial X_0} \frac{\partial}{\partial f^*} N(f^*) \\
&= e^{-\alpha T} \frac{\partial f^*}{\partial X_0} n(f^*) \\
&= \frac{\alpha e^{-2\alpha T}}{\sigma\sqrt{T}(1 - e^{-\alpha T})} n(f^*).
\end{aligned}$$

Similar results can be obtained for put option sensitivities.

# Bibliography

[1] Nielsen, L. T. *Pricing and Hedging of Derivative Securities.* Oxford University Press, 1999.

[2] Elliott, R. J. and Kopp, P. E. *Mathematics of Financial Markets*, Second Edition. Springer, 2004.